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AVERAGED ROTATIONS AT FINITE PLANE STRAIN

V. D. Bondar'

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The averaged rotations and other mechanical parameters at finite plane strains of an elastic material, which are characterized by a linear relation between the Cauchy stresses and the Almansi strains, are studied. The form of the elastic potential is determined. The displacement problem is reduced to a boundary-value problem for complex potentials, which is solved in terms of Cauchy-type integrals for the specified boundary displacements. The results obtained are compared with the linear solution.

1. In the linear theory of elasticity, the strain and rotation of one element of a material, which are defined as the symmetric and antisymmetric parts of the displacement gradient, are small quantities. In the nonlinear theory of elasticity, the strains and rotations are finite. In actual-state variables, the strains are characterized by the Almansi tensor and, according to V. V. Novozhilov, the rotations are characterized by the rotations averaged over the elementary volume. We consider the behavior of the averaged rotations and other mechanical parameters of an elastic body under plane strain within the framework of the nonlinear theory of elasticity.

Static deformation is governed by the equations of equilibrium, the continuity equation, Murnaghan's law, the strain–displacement relations, and the boundary conditions

$$\operatorname{div} P + \rho \boldsymbol{f} = 0, \qquad \nu = \rho/\rho_0 = \sqrt{1 - 2\varepsilon_1 + 4\varepsilon_2 - 8\varepsilon_3},$$

$$\varepsilon_1 = \operatorname{tr}\varepsilon, \qquad 2\varepsilon_2 = (\operatorname{tr}\varepsilon)^2 - \operatorname{tr}\varepsilon^2, \quad \varepsilon_3 = \det\varepsilon, \tag{1}$$

$$P = \nu(G - 2\varepsilon)\cdot\frac{d\Phi}{d\varepsilon}, \quad 2\varepsilon = \nabla\boldsymbol{u} + \boldsymbol{u}\nabla - (\nabla\boldsymbol{u})\cdot(\boldsymbol{u}\nabla), \quad \boldsymbol{u}\Big|_{\Sigma_u} = \boldsymbol{h}, \quad P\cdot\boldsymbol{n}\Big|_{\Sigma_p} = \boldsymbol{p}.$$

Here $\boldsymbol{u}$, $\boldsymbol{f}$, $\boldsymbol{h}$, $\boldsymbol{p}$, and $\boldsymbol{n}$ are the vectors of displacements, body forces, boundary displacements, stresses, and the outward normal, respectively, G, P, and ε are the metric, Cauchy stress, and Almansi strain tensors, respectively, $\nabla\boldsymbol{u}$ and $\boldsymbol{u}\nabla$ are the displacement and transposed displacement gradients, respectively, ε_1, ε_2, ε_3, ρ_0, ρ, ν, and Φ are the basis strain invariants, the initial, actual, and relative densities of the material, and the elastic potential, respectively, and Σ_u and Σ_p are the regions on the surface of the body where the displacements and stresses are specified, respectively [1].

For a homogeneous isotropic body under plane strain, the conditions $\varepsilon_3 = 0$ and $\Phi = \Phi(\varepsilon_1, \varepsilon_2)$ hold. With allowance for the expressions for the tensor gradients of the strain invariants and the Hamilton–Kelly identity for strains [1]

$$\frac{d\varepsilon_1}{d\varepsilon} = G, \quad \frac{d\varepsilon_2}{d\varepsilon} = \varepsilon_1 G - \varepsilon, \quad \varepsilon^2 - \varepsilon_1\varepsilon + \varepsilon_2 G = 0,$$

Murnaghan's law takes the form of a quasilinear stress–strain relation, which can be written in the form of Hooke's law with the variable coefficients of elasticity expressed in terms of the elastic potential:

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$$P = \varepsilon_1 \Lambda(\varepsilon_1, \varepsilon_2) G + 2M(\varepsilon_1, \varepsilon_2)\varepsilon; \tag{2}$$

$$\varepsilon_1 \Lambda = \nu\Big(\frac{\partial \Phi}{\partial \varepsilon_1} + (\varepsilon_1 - 2\varepsilon_2)\frac{\partial \Phi}{\partial \varepsilon_2}\Big), \quad 2M = -\nu\Big(2\frac{\partial \Phi}{\partial \varepsilon_1} + \frac{\partial \Phi}{\partial \varepsilon_2}\Big), \quad \nu = \sqrt{1 - 2\varepsilon_1 + 4\varepsilon_2}. \tag{3}$$

Transforming formulas (3), we obtain the potential gradients

$$\nu^3 \frac{\partial \Phi}{\partial \varepsilon_1} = \varepsilon_1 \Lambda + (2\varepsilon_1 - 4\varepsilon_2)M = \nu^3 \Phi_1, \quad \nu^3 \frac{\partial \Phi}{\partial \varepsilon_2} = -2(\varepsilon_1 \Lambda + M) = \nu^3 \Phi_2, \tag{4}$$

and the compatibility condition $\partial \Phi_1/\partial \varepsilon_2 = \partial \Phi_2/\partial \varepsilon_1$ imposes a restriction on the coefficients of elasticity:

$$2\frac{\partial(\varepsilon_1 \Lambda + M)}{\partial \varepsilon_1} + \frac{\partial(\varepsilon_1 \Lambda + (2\varepsilon_1 - 4\varepsilon_2)M)}{\partial \varepsilon_2} + 6M = 0. \tag{5}$$

From the coefficients of elasticity subject to condition (5), we find the potential gradients (4) and, hence, determine the elastic potential by the quadrature [2]

$$\Phi(\varepsilon_1, \varepsilon_2) = \int_0^{\varepsilon_1} \Phi_1(\varepsilon_1, \varepsilon_2)\, d\varepsilon_1 + \int_0^{\varepsilon_2} \Phi_2(0, \varepsilon_2)\, d\varepsilon_2 + \text{const}. \tag{6}$$

In the absence of body forces, relations (1) can be written in the complex actual-state variables $z = x + iy$ and $\bar{z} = x - iy$ (x and y are the Cartesian coordinates) to give the main plane-strain problem

$$\begin{gathered}
\frac{\partial P^{11}}{\partial z} + \frac{\partial P^{12}}{\partial \bar{z}} = 0, \quad \nu = \sqrt{1 - 2\varepsilon_1 + 4\varepsilon_2}, \quad \varepsilon_1 = \varepsilon^{12}, \quad 4\varepsilon_2 = (\varepsilon^{12})^2 - \varepsilon^{11}\varepsilon^{22}, \\
P^{11} = \bar{P}^{22} = 2M(\varepsilon_1, \varepsilon_2)\varepsilon^{11}, \quad P^{12} = 2N(\varepsilon_1, \varepsilon_2)\varepsilon^{12} \quad (N = \Lambda + M), \\
\varepsilon^{11} = \bar{\varepsilon}^{22} = 2\frac{\partial u}{\partial \bar{z}}\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big), \quad 1 - \varepsilon^{12} = \Big(1 - \frac{\partial u}{\partial z}\Big)\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big) + \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{u}}{\partial z}, \\
u\Big|_{L_u} = h(s), \qquad P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\Big|_{L_p} = 2ip(s).
\end{gathered} \tag{7}$$

Here L_u and L_p are the parts of the contour L [the boundary of the cross section of the body D parallel to the deformation plane determined by the equations $z = z(s)$ and $\bar{z} = \bar{z}(s)$], where the displacements and stresses are specified, respectively, s is the arc of the contour, and the numerical superscripts denote the complex vector and tensor components; the complex displacement-vector and stress-tensor components are related to the Cartesian components of the same quantities (denoted by alphabetical subscripts) by the following component-transformation formulas [3]:

$$u^1 = u = u_x + iu_y, \qquad u^2 = \bar{u} = u_x - iu_y,$$

$$P^{11} = P_{xx} - P_{yy} + 2iP_{xy}, \qquad P^{22} = P_{xx} - P_{yy} - 2iP_{xy}, \qquad P^{12} = P^{21} = P_{xx} + P_{yy}.$$

When only the displacements are specified on the boundary, we have the displacement problem

$$\frac{\partial}{\partial z}\Big[2M\frac{\partial u}{\partial \bar{z}}\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big)\Big] + \frac{\partial}{\partial \bar{z}}\Big[N\Big(1 - \Big(1 - \frac{\partial u}{\partial z}\Big)\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big) - \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{u}}{\partial z}\Big)\Big] = 0, \quad u\Big|_L = h(s), \tag{8}$$

which follows from (7). The displacements found from (8) determine the density and the stresses:

$$\begin{gathered}
u = u(z, \bar{z}), \qquad \nu = \Big(1 - \frac{\partial u}{\partial z}\Big)\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big) - \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{u}}{\partial z}, \\
P^{11} = \bar{P}^{22} = 4M\frac{\partial u}{\partial \bar{z}}\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big), \quad P^{12} = 2N\Big[1 - \Big(1 - \frac{\partial u}{\partial z}\Big)\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big) - \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{u}}{\partial z}\Big].
\end{gathered} \tag{9}$$

According to V. V. Novozhilov [4], the rotation of an elementary volume at finite strains can be described by the volume-average rotations $\Omega^{\alpha\beta}$, which are expressed in terms of the linear rotations $\omega^{\alpha\beta}$ and strains $e^{\alpha\beta}$. In the plane problem, the linear quantities, their invariants, and the averaged rotations are given by

$$\begin{gathered}
\omega^{11} = \bar{\omega}^{22} = 0, \quad \omega^{21} = \bar{\omega}^{12} = \frac{\partial u}{\partial z} - \frac{\partial \bar{u}}{\partial \bar{z}}, \quad \omega_1 = 0, \quad 4\omega_2 = \omega^{12}\omega^{21}, \\
e^{11} = \bar{e}^{22} = 2\,\frac{\partial u}{\partial \bar{z}}, \quad e^{12} = e^{21} = \frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}}, \quad e_1 = e^{12}, \quad 4e_2 = e^{12}e^{21} - e^{11}e^{22}, \\
4(1 - e_1 + e_2) = (2 - e^{12})^2 - e^{11}e^{22} = 4\nu + \Big(\frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}}\Big)^2 - 4\,\frac{\partial u}{\partial z}\,\frac{\partial \bar{u}}{\partial \bar{z}} = 4\nu + (\omega^{21})^2, \\
\Omega^{11} = \bar{\Omega}^{22} = 0, \quad \Omega^{21} = \frac{\omega^{21}}{\sqrt{1 - e_1 + e_2}} = \frac{2\omega^{21}}{\sqrt{4\nu + (\omega^{21})^2}} \quad (\Omega^{21} = 2i\Omega_{xy}, \quad \omega^{21} = 2i\omega_{xy}).
\end{gathered} \tag{10}$$

2. We consider the plane strain of materials that are characterized by a linear relation between the Cauchy and Almansi tensors. It follows from Murnaghan's law (2) and condition (5) that the coefficients of elasticity are constant and related by the condition

$$\Lambda = \text{const}, \qquad M = \text{const}, \qquad N = \Lambda + M = 0. \tag{11}$$

In this case, the law (2) and the potential (6) include only one constant:

$$P = M(2\varepsilon - \varepsilon_1 G), \quad P^{11} = \bar{P}^{22} = 2M\varepsilon^{11}, \quad P^{12} = 0, \quad \Phi(\varepsilon_1, \varepsilon_2) = \frac{M(1 - \varepsilon_1)}{\sqrt{1 - 2\varepsilon_1 + 4\varepsilon_2}} - M;$$

Eq. (8) takes the form

$$\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big)4\,\frac{\partial^2 u}{\partial z\,\partial \bar{z}} - \frac{\partial u}{\partial \bar{z}}\,4\,\frac{\partial^2 \bar{u}}{\partial z\,\partial \bar{z}} = 0.$$

Supplementing this equation by the complex-conjugate equality and expressing the Laplace operator in the complex variables $\Delta = \partial_{xx} + \partial_{yy} = 4\partial_{z\bar{z}}$, we obtain the homogeneous algebraic system of equations for Δu and $\Delta \bar{u}$

$$\Big(1 - \frac{\partial \bar{u}}{\partial \bar{z}}\Big)\Delta u - \frac{\partial u}{\partial \bar{z}}\,\Delta \bar{u} = 0, \qquad -\frac{\partial \bar{u}}{\partial z}\,\Delta u + \Big(1 - \frac{\partial u}{\partial z}\Big)\Delta \bar{u} = 0.$$

The determinant of this system is equal to the relative density (9) and, hence, it does not vanish; therefore, the system has only the trivial solution $\Delta u = 0$ and $\Delta \bar{u} = 0$. In this case, problem (8) takes the form of the Dirichlet problem for the harmonic equation:

$$\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0, \qquad u\Big|_L = h(s). \tag{12}$$

Using (9)–(12), we express the quantities considered in terms of two complex potentials $\varphi(z)$ and $\psi(z)$ and obtain the boundary-value problem for the potentials:

$$\begin{gathered}
u(z, \bar{z}) = \varphi(z) - \bar{\psi}(\bar{z}), \qquad \nu(z, \bar{z}) = (1 - \varphi'(z))(1 - \bar{\varphi}'(\bar{z})) - \psi'(z)\bar{\psi}'(\bar{z}), \\
P^{11}(\bar{z}) = -4M\bar{\psi}'(\bar{z})(1 - \bar{\varphi}'(\bar{z})), \qquad P^{22}(z) = -4M\psi'(z)(1 - \varphi'(z)), \qquad P^{12} = 0, \\
\Omega^{11} = \bar{\Omega}^{22} = 0, \qquad \Omega^{21}(z, \bar{z}) = \frac{2(\varphi'(z) - \bar{\varphi}'(\bar{z}))}{\sqrt{4\nu + (\varphi'(z) - \bar{\varphi}'(\bar{z}))^2}};
\end{gathered} \tag{13}$$

$$\varphi(z) - \bar{\psi}(\bar{z}) = h(s). \tag{14}$$

When only the stresses are specified on the boundary, the potentials are determined from the second condition in (7):

$$\bar{\psi}(\bar{z}) - \int \bar{\psi}'(\bar{z})\bar{\varphi}'(\bar{z})\,d\bar{z}\Big|_L = g(s), \quad g(s) = \frac{1}{2M}\Big(i\int_0^s p(s)\,ds + C\Big), \quad C = \text{const}. \tag{15}$$

Thus, for the materials considered, the potential representations are linear for displacements and nonlinear for other quantities. As a result, the boundary-value problem for potentials is linear in displacements and nonlinear in stresses.

3. To compare the above results with those obtained within the framework of the linear theory, we introduce the potentials $\varphi_1 = 4M^2\varphi$ and $\psi_1 = 2M\psi$. As a result, formulas (13)–(15) become

$$2Mu = \frac{\varphi_1}{2M} - \bar{\psi}_1, \quad \nu = \Big(1 - \frac{\varphi_1'}{4M^2}\Big)\Big(1 - \frac{\bar{\varphi}_1'}{4M^2}\Big) - \frac{\psi_1'\bar{\psi}_1'}{4M^2}, \quad P^{11} = \bar{P}^{22} = -2\bar{\psi}_1'\Big(1 - \frac{\bar{\varphi}_1'}{4M^2}\Big),$$

$$P^{12} = 0, \qquad \Omega^{11} = \bar{\Omega}^{22} = 0, \qquad \Omega^{21} = \frac{2(\varphi_1' - \bar{\varphi}_1')}{\sqrt{64\nu M^4 - (\varphi_1' - \bar{\varphi}_1')^2}}, \tag{16}$$

$$\frac{\varphi_1}{2M} - \bar{\psi}_1\Big|_{L_u} = 2Mh, \qquad \bar{\psi}_1 - \frac{1}{4M^2}\int \bar{\psi}_1'\bar{\varphi}_1'\,d\bar{z}\Big|_{L_p} = i\int_0^s p\,ds + C.$$

Let P_0 and L_0 be the characteristic stress and dimension, respectively and $\sigma = P_0/M$ be a dimensionless parameter. Expressing the quantities considered in terms of dimensionless quantities (asterisked)

$$P^{11} = P_0P^{11}_*, \quad P^{12} = P_0P^{12}_*, \quad p = P_0p_*, \quad M = P_0M_*, \quad u = L_0u_*,$$

$$z = L_0z_*, \quad s = L_0s_*, \quad h = L_0h_*, \quad \sigma = 1/M_*, \quad \varphi_1 = P_0^2L_0\varphi_{1*},$$

$$\psi_1 = P_0L_0\psi_{1*}, \quad C = P_0L_0C_*, \quad \nu = \nu_*, \quad \Omega^{21} = \Omega^{21}_*,$$

we write relations (16) in dimensionless form:

$$2M_*u_* = \frac{\sigma}{2}\varphi_{1*} - \bar{\psi}_{1*}, \quad \nu_* = \Big(1 - \frac{\sigma^2}{4}\varphi_{1*}'\Big)\Big(1 - \frac{\sigma^2}{4}\bar{\varphi}_{1*}'\Big) - \frac{\sigma^2}{4}\psi_{1*}'\bar{\psi}_{1*}',$$

$$P^{11}_* = \bar{P}^{22}_* = -2\bar{\psi}_{1*}'\Big(1 - \frac{\sigma^2}{4}\bar{\varphi}_{1*}'\Big), \quad P^{12}_* = 0,$$

$$\Omega^{11}_* = \bar{\Omega}^{22}_* = 0, \qquad \Omega^{12}_* = \frac{2\sigma^2(\varphi_{1*}' - \bar{\varphi}_{1*}')}{\sqrt{64\nu_* + \sigma^4(\varphi_{1*}' - \bar{\varphi}_{1*}')^2}}, \tag{17}$$

$$\frac{\sigma}{2}\varphi_{1*} - \bar{\psi}_{1*}\Big|_{L_u} = 2M_*h_*, \qquad \bar{\psi}_{1*} - \frac{\sigma^2}{4}\int \bar{\psi}_{1*}'\bar{\varphi}_{1*}'\,d\bar{z}_*\Big|_{L_p} = i\int_0^{s_*} p_*\,ds_* + C_*.$$

Assuming that the dimensionless quantities have finite moduli in the closed region and the dimensionless parameter is small compared to unity, we can ignore small (parameter-containing) terms in (17). As a result (after reverting to the dimensional quantities), we obtain the formulas

$$2Mu = -\bar{\psi}_1, \quad P^{11} = \bar{P}^{22} = -2\bar{\psi}_1', \quad P^{12} = 0, \quad \Omega^{11} = \bar{\Omega}^{22} = 0, \quad \Omega^{21} = 0,$$

$$\nu = 1, \quad -\bar{\psi}_1\Big|_{L_u} = 2Mh, \qquad \bar{\psi}_1\Big|_{L_p} = i\int_0^s p\,ds + C,$$

which coincide with the formulas of linear elasticity [5, 6]

$$2\mu u = (3 - 4\nu)\varphi_1 - z\bar{\varphi}_1' - \bar{\psi}_1, \quad \mu\omega^{11} = \mu\bar{\omega}^{22} = 0, \quad \mu\omega^{21} = 2(1 - \nu)(\varphi_1' - \bar{\varphi}_1'),$$

$$P^{11} = \bar{P}^{22} = -2(z\bar{\varphi}_1'' + \bar{\psi}_1'), \quad P^{12} = 2(\varphi_1' + \bar{\varphi}_1'), \quad \nu = 1 - \frac{1 - 2\nu}{\mu}(\varphi_1' + \bar{\varphi}_1'),$$

$$(3 - 4\nu)\varphi_1 - z\bar{\varphi}_1' - \bar{\psi}_1\Big|_{L_u} = 2\mu h, \qquad \varphi_1 + z\bar{\varphi}_1' + \bar{\psi}_1\Big|_{L_p} = i\int_0^s p\,ds + C$$

(μ and ν are the shear modulus and Poisson's ratio, respectively) when the potential φ_1 is equal to zero.

4. Let l be a closed contour in the region $\bar{D}$ that is determined by the equations $z = z(s)$ and $\bar{z} = \bar{z}(s)$. If the stress vector is specified at each point of the contour, the components of the principal vector F and the principal moment M of the contour forces are given by

$$F = F_x + iF_y = \oint p\,ds = \frac{1}{2i}\oint \Big(P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\Big)ds,$$

$$M = \mathrm{Re}\Big\{-i\oint \bar{z}p\,ds\Big\} = \mathrm{Re}\Big\{-\frac{1}{2}\oint \bar{z}\Big(P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\Big)ds\Big\}.$$

Using relations (13), we write these formulas in the form

$$\bar{F} = 2Mi[\psi(z) - \chi(z)]_l, \quad M = -2M\mathrm{Re}\,[z(\psi(z) - \chi(z))]_l, \quad \chi = \int \psi'\varphi'\,dz, \tag{18}$$

where $[A]_l$ denotes the increment in A for the positive path tracing of the contour l.

Formulas (13) imply that the stresses, rotations, and density preserve their values when φ and ψ are replaced by the potentials $\varphi + \alpha$ and $\psi + \beta$ (α and β = const). Owing to this arbitrariness, one can fix the values of the potentials at one of the points of the region. If, along with the indicated quantities, the displacements must also preserve their values, then the constants must satisfy the condition $\alpha - \bar{\beta} = 0$. In this case, only one potential can be fixed at a certain point.

The complex potentials are multivalued in the infinite, simply connected region D with boundary L. Let the quantities considered be uniquely determined in D. In accordance with (13), for any closed contour $l \in \bar{D}$ enclosing L, we have

$$[\varphi - \bar{\psi}]_l = 0, \quad [(1-\varphi')(1-\bar{\varphi}') - \psi'\bar{\psi}']_l = 0, \quad [\psi'(1-\varphi')]_l = 0, \quad \Big[\frac{\varphi' - \bar{\varphi}'}{\sqrt{4\nu + (\varphi' - \bar{\varphi}')^2}}\Big]_l = 0.$$

With allowance for the properties of the increments in the function $W(z,\bar{z})$ [7]

$$\frac{\partial}{\partial z}[W(z,\bar{z})]_l = \Big[\frac{\partial W(z,\bar{z})}{\partial z}\Big]_l, \qquad \frac{\partial}{\partial \bar{z}}[W(z,\bar{z})]_l = \Big[\frac{\partial W(z,\bar{z})}{\partial \bar{z}}\Big]_l,$$

it follows that the potential gradients must be single-valued: $[\varphi']_l = 0$ and $[\psi']_l = 0$. The potentials can be expressed in terms of the single-valued functions φ^0 and ψ^0, and their increments are related by the condition

$$[\varphi]_l = 2\pi i a, \quad [\psi]_l = 2\pi i b, \quad a + \bar{b} = 0, \quad \varphi(z) = a\ln z + \varphi^0(z), \quad \psi(z) = b\ln z + \psi^0(z). \tag{19}$$

Expanding φ^0 and ψ^0 in the Laurent series and calculating the quantities (13), we infer that the stresses, rotations, and density are bounded in the infinite region, provided the potentials have the form

$$\varphi(z) = a\ln z + a_1 z + \varphi_0(z), \qquad \psi(z) = b\ln z + b_1 z + \psi_0(z),$$
$$\varphi_0(z) = \sum_{n=0}^{\infty} a_{-n}z^{-n}, \quad \psi_0(z) = \sum_{n=0}^{\infty} b_{-n}z^{-n}. \tag{20}$$

Let the values P^{11}_∞, P^{12}_∞, ν_∞, and Ω^{21}_∞ be specified at infinity. Then, in accordance with (13) and (20), we obtain the relations

$$P^{11}_\infty = -4M\bar{b}_1(1-\bar{a}_1), \quad P^{12}_\infty = 0, \quad \nu_\infty = |1-a_1|^2 - |b_1|^2, \quad \Omega^{21}_\infty\sqrt{4\nu_\infty + (a_1 - \bar{a}_1)^2} = 2(a_1 - \bar{a}_1).$$

Assuming that $a_1 = a_{1x} + ia_{1y}$ and $b_1 = b_{1x} + ib_{1y}$ and using the relations between the complex and Cartesian stress and rotation components, we obtain the following equations for the second coefficients in expansions (20):

$$-P^\infty_{xx} = 2M(b_{1x}(1-a_{1x}) + b_{1y}a_{1y}), \qquad P^\infty_{xy} = 2M(-b_{1x}a_{1y} + b_{1y}(1-a_{1x})),$$
$$(\Omega^\infty_{xy})^2(\nu_\infty - a_{1y}^2) = a_{1y}^2, \qquad 4M^2|1-a_1|^4 - 4M^2\nu_\infty|1-a_1|^2 - |P^\infty_{xx} - iP^\infty_{xy}|^2 = 0. \tag{21}$$

Hence,

$$(1-a_{1x})^2 = \frac{\nu_\infty}{2}\frac{1-(\Omega_{xy}^\infty)^2}{1+(\Omega_{xy}^\infty)^2} + \frac{1}{2M}\sqrt{M^2\nu_\infty^2 + (P_{xx}^\infty)^2 + (P_{xy}^\infty)^2}, \quad a_{1y}^2 = \frac{\nu_\infty(\Omega_{xy}^\infty)^2}{1+(\Omega_{xy}^\infty)^2},$$

$$2Mb_{1x} = -\frac{(1-a_{1x})P_{xx}^\infty + a_{1y}P_{xy}^\infty}{(1-a_{1x})^2 + a_{1y}^2}, \qquad 2Mb_{1y} = \frac{(1-a_{1x})P_{xy}^\infty - a_{1y}P_{xx}^\infty}{(1-a_{1x})^2 + a_{1y}^2}.$$

The conditions $P_\infty^{12} = 0$ and $(1-a_{1x})^2 \geqslant 0$ impose restrictions on the specified quantities:

$$P_{xx}^\infty + P_{yy}^\infty = 0, \quad M\nu_\infty(1-(\Omega_{xy}^\infty)^2) + (1+(\Omega_{xy}^\infty)^2)\sqrt{M^2\nu_\infty^2 + (P_{xx}^\infty)^2 + (P_{xy}^\infty)^2} \geqslant 0.$$

Calculating the principal contour-force vector (18) with allowance for (20) and using (19), we obtain the equalities

$$\bar{F} = 4\pi M(b_1 a - (1-a_1)b), \qquad a + \bar{b} = 0 \tag{22}$$

that determine the coefficients a and b:

$$a = \frac{(1-a_1)F - \bar{b}_1\bar{F}}{4\pi M\nu_\infty}, \qquad b = \frac{b_1 F - (1-\bar{a}_1)\bar{F}}{4\pi M\nu_\infty}.$$

Thus, in the expansions of the potentials (20), the first pairs of coefficients are determined by the elastic properties of the material, the contour and peripheral loads, and the density and rotation values at the periphery.

For the adopted conditions, displacement (13), which corresponds to potentials (20),

$$u = a_1 z - \bar{b}_1\bar{z} - \bar{b}\ln(z\bar{z}) + \sum_{n=0}^{\infty}(a_{-n}z^{-n} - \bar{b}_{-n}\bar{z}^{-n})$$

unlimitedly increases at infinity. For this displacement to be limited, one should also set $a = -\bar{b} = 0$ and $a_1 = b_1 = 0$; by virtue of (21) and (22), this constrains the quantities at the contour and periphery:

$$F_x = F_y = 0, \quad P_{xx}^\infty = P_{xy}^\infty = 0, \quad \Omega_{xy}^\infty = 0, \quad \nu_\infty = 1.$$

If all the mechanical quantities are limited in the infinite region, potentials (20) become single-valued functions:

$$\varphi(z) = \sum_{n=0}^{\infty} a_{-n}z^{-n}, \qquad \psi(z) = \sum_{n=0}^{\infty} b_{-n}z^{-n}.$$

If the potentials have the form (20) in this region, we can consider the boundary-value problem for the single-valued potentials φ_0 and ψ_0, which follows from (14):

$$\varphi_0(z) - \bar{\psi}_0(\bar{z})\Big|_L = h_0(s), \quad h_0(s) = h(s) + \bar{b}_1\bar{z}(s) - a_1 z(s) + \bar{b}\ln|z(s)|^2, \quad z \in L. \tag{23}$$

Since problem (14) is similar to (23), below we consider the first problem.

5. We map conformally the simply connected region D onto the interior of the unit circle K with circumference γ:

$$z = w(\zeta), \qquad w'(\zeta) \neq 0, \qquad \zeta = r\exp(i\theta) \in K.$$

As a result, the complex potentials take the form

$$\varphi(z) = \varphi(\zeta), \quad \varphi'(z) = \frac{\varphi'(\zeta)}{w'(\zeta)}, \qquad \psi(z) = \psi(\zeta), \quad \psi'(z) = \frac{\psi'(\zeta)}{w'(\zeta)},$$

the mechanical quantities (13) are given by

$$u = \varphi(\zeta) - \bar{\psi}(\bar{\zeta}), \quad \Omega^{11} = \bar{\Omega}^{22} = 0, \quad \Omega^{21} = \frac{2(\varphi'(\zeta)\bar{w}'(\bar{\zeta}) - \bar{\varphi}'(\bar{\zeta})w'(\zeta))}{\sqrt{4\nu|w'(\zeta)|^4 + (\varphi'(\zeta)\bar{w}'(\bar{\zeta}) - \bar{\varphi}'(\bar{\zeta})w'(\zeta))^2}},$$

$$P^{11} = \bar{P}^{22} = -4M\frac{\bar{\psi}'(\bar{\zeta})(\bar{w}'(\bar{\zeta}) - \bar{\varphi}'(\bar{\zeta}))}{(\bar{w}'(\bar{\zeta}))^2}, \quad P^{12} = 0, \quad \nu = \frac{|w'(\zeta) - \varphi'(\zeta)|^2 - |\psi'(\zeta)|^2}{|w'(\zeta)|^2}, \tag{24}$$

and condition (14) becomes a boundary condition for the potentials in the unit circle:

$$\varphi(\tau) - \bar{\psi}(\bar{\tau}) = h(\tau), \qquad \tau = \exp(i\theta) \in \gamma. \tag{25}$$

Let the boundary-displacement function belong to the class of Hölder functions and one of the potentials vanish at the center of the circle:

$$h(\tau) \in H, \qquad \tau \in \gamma, \qquad \psi(0) = 0. \tag{26}$$

Multiplying (25) by $1/(2\pi i(\tau-\zeta))$, integrating over the circumference, and taking into account the well-known properties of the integrals [6]

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{\varphi(\tau)\,d\tau}{\tau-\zeta} = \varphi(\zeta), \qquad \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\bar{\psi}(\bar{\tau})\,d\tau}{\tau-\zeta} = \bar{\psi}(0) = 0 \quad (\zeta \in K),$$

we express the first potential in the form

$$\varphi(\zeta) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{h(\tau)\,d\tau}{\tau-\zeta} \qquad (\zeta \in K). \tag{27}$$

Passing to conjugate quantities, from condition (25) we obtain the expression for the second potential:

$$\psi(\zeta) = \bar{\varphi}(0) - \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\bar{h}(\bar{\tau})\,d\tau}{\tau-\zeta} \qquad (\zeta \in K).$$

After the constant $\bar{\varphi}(0)$ is determined from the condition $\psi(0) = 0$, the second potential takes the form

$$\psi(\zeta) = \frac{1}{2\pi i}\int\limits_{\gamma} \bar{h}(\bar{\tau})\,\frac{d\tau}{\tau} - \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\bar{h}(\bar{\tau})\,d\tau}{\tau-\zeta}. \tag{28}$$

Formulas (27) and (28) give the solution of problem (25). Indeed, when ζ tends from the circle to a certain boundary point $\tau_0 = \exp(i\theta_0)$, the potentials take on definite limit values by virtue of (26); according to the Sochotsky–Plemelj formulas [6], the latter have the form

$$\varphi_+(\tau_0) \equiv \varphi(\tau_0) = \frac{h(\tau_0)}{2} + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{h(\tau)\,d\tau}{\tau-\tau_0} = \frac{h(\tau_0)}{2} + \frac{1}{2\pi}\int\limits_{0}^{2\pi} \frac{h(\theta)\,d\theta}{1-\exp(i(\theta_0-\theta))},$$

$$\psi_+(\tau_0) \equiv \psi(\tau_0) = \frac{1}{2\pi i}\int\limits_{\gamma} \bar{h}(\bar{\tau})\,\frac{d\tau}{\tau} - \frac{\bar{h}(\bar{\tau}_0)}{2} - \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\bar{h}(\bar{\tau})\,d\tau}{\tau-\tau_0} = -\frac{\bar{h}(\bar{\tau}_0)}{2} - \frac{1}{2\pi}\int\limits_{0}^{2\pi} \frac{\bar{h}(\theta)\,d\theta}{\exp(i(\theta-\theta_0))-1}.$$

It follows that the difference $\varphi(\tau_0) - \bar{\psi}(\bar{\tau}_0)$ is equal to $h(\tau_0)$. This means that the potentials satisfy the boundary condition (25).

6. Let an infinite plate in the actual state have an elliptic hole with semiaxes a and b $(a > b)$. We consider its deformation for the case where the displacements are equal to the specified values on the hole boundary and vanish at infinity. We assume that the Cartesian axes coincide with the ellipse axes. The conformal mapping of the exterior of the ellipse D onto the interior of the unit circle K (the point $z = \infty$ corresponds to the point $\zeta = 0$) is defined by the function [6]

$$z = w(\zeta) = n\Big(m\zeta + \frac{1}{\zeta}\Big), \quad n = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b}, \quad \zeta = r\exp(i\theta) \in K. \tag{29}$$

The parameters $0 \leqslant m < 1$ and $0 < n < \infty$ characterize the shape and dimensions of the ellipse, respectively [for $m = 1$, the ellipse becomes a cut, and the mapping is not conformal since $w'(\pm 1) = 0$]. One can readily establish the formulas

$$x = n\Big(mr + \frac{1}{r}\Big)\cos\theta, \qquad y = n\Big(mr - \frac{1}{r}\Big)\sin\theta,$$

which determine the elliptic coordinates r and θ in the plane of the circle. In these coordinates, the equations of the boundary ellipse L have the form

$$x_L = n(m+1)\cos\theta, \qquad y_L = n(m-1)\sin\theta. \tag{30}$$

Let the boundary displacements depend on the dimensions and shape of the hole and the positive parameter α:

$$h_x = n(m\cos\theta + \alpha\sin\theta), \quad h_y = n(m\sin\theta + \alpha\cos\theta), \quad r = 1$$
$$(h = h_x + ih_y = n(m\tau - \bar{i}\alpha\bar{\tau}), \quad \tau = \exp(i\theta) \in \gamma). \tag{31}$$

In this case, we have $h(\tau) \in H$; therefore, integrals (27) and (28) determine the complex potentials in the form

$$\varphi(z) = nm\zeta, \qquad \psi(\zeta) = in\alpha\zeta.$$

For these potentials and mapping (29), formulas (24) become

$$u = n(m\zeta + i\alpha\bar{\zeta}), \qquad \Omega^{11} = \bar{\Omega}^{22} = 0,$$

$$\Omega^{21} = \frac{-2m(\zeta^2 - \bar{\zeta}^2)}{\sqrt{4(1-\alpha^2\zeta^2\bar{\zeta}^2)(m\zeta^2-1)(m\bar{\zeta}^2-1) + m^2(\zeta^2-\bar{\zeta}^2)^2}}, \tag{32}$$

$$P^{11} = \bar{P}^{22} = -\frac{4i\alpha M\bar{\zeta}^2}{(m\bar{\zeta}^2-1)^2}, \quad P^{12} = 0, \quad \nu = \frac{1-\alpha^2\zeta^2\bar{\zeta}^2}{(m\zeta^2-1)(m\bar{\zeta}^2-1)}.$$

By virtue of (30) and (31), in the initial state of the plate the coordinates x_0 and y_0 of the points on the hole contour L_0 are given by

$$x_0 = x_L - h_x = n(\cos\theta - \alpha\sin\theta), \quad y_0 = y_L - h_y = -n(\sin\theta + \alpha\cos\theta).$$

Elimination of the parameter θ leads to the contour equation $x_0^2 + y_0^2 = R_0^2$, where $R_0 = n\sqrt{1+\alpha^2}$. Hence, the hole in the plate is circular before deformation.

The density and the displacement, stress, and averaged-rotation components, which are determined in the elliptic coordinates r and θ in terms of the quantities (32) and mapping (29) by the expressions [7]

$$\nu(\zeta,\bar{\zeta}) = \nu(r,\theta), \qquad u_r + iu_\theta = \frac{\bar{\zeta}\bar{w}'(\bar{\zeta})}{|\zeta|\,|w'(\zeta)|}\,u,$$

$$P_{rr} - P_{\theta\theta} + 2iP_{r\theta} = \frac{\bar{\zeta}\bar{w}'(\bar{\zeta})}{\zeta w'(\zeta)}\,P^{11}, \qquad P_{rr} + P_{\theta\theta} = P^{12},$$

$$\Omega_{rr} - \Omega_{\theta\theta} + i(\Omega_{r\theta} + \Omega_{\theta r}) = \frac{\bar{\zeta}\bar{w}'(\bar{\zeta})}{\zeta w'(\zeta)}\,\Omega^{11}, \quad \Omega_{rr} + \Omega_{\theta\theta} + i(\Omega_{r\theta} - \Omega_{\theta r}) = \Omega^{21},$$

have the form

$$u_r = mnr\,\frac{mr^2 + \alpha r^2\sin 2\theta - \cos 2\theta}{\sqrt{1 + m^2r^4 - 2mr^2\cos 2\theta}}, \qquad u_\theta = -nr\,\frac{\alpha - \alpha mr^2\cos 2\theta + m\sin 2\theta}{\sqrt{1 + m^2r^4 - 2mr^2\cos 2\theta}},$$

$$\nu = \frac{1-\alpha^2r^4}{1 + m^2r^4 - 2mr^2\cos 2\theta}, \quad P_{rr} = P_{\theta\theta} = 0, \quad P_{r\theta} = -\frac{2\alpha Mr^2}{1 + m^2r^4 - 2mr^2\cos 2\theta},$$
$$\Omega_{rr} = \Omega_{\theta\theta} = 0, \qquad \Omega_{r\theta} = -\frac{mr^2\sin 2\theta}{\sqrt{a\cos^2 2\theta + 2b\cos 2\theta + c}}, \tag{33}$$
$$a = m^2r^4, \quad b = -mr^2(1-\alpha^2r^4), \qquad c = 1 - \alpha^2r^4(1+m^2r^4).$$

By virtue of the condition

$$1 + m^2 r^4 - 2mr^2 \cos 2\theta \geqslant (1 - mr^2)^2 > 0, \tag{34}$$

the displacements are real. The stress field is such that at the sites perpendicular to the coordinate lines, only the shear stresses act. It follows from (33) that at infinity ($r = 0$), the displacements, stresses, and rotations decrease unlimitedly and the relative density tends to unity:

$$u_r^\infty = u_\theta^\infty = 0, \quad P_{rr}^\infty = P_{\theta\theta}^\infty = P_{r\theta}^\infty = 0, \quad \Omega_{rr}^\infty = \Omega_{\theta\theta}^\infty = \Omega_{r\theta}^\infty = 0, \quad \nu^\infty = 1.$$

As the boundary ellipse $L(r = 1)$ is approached, these quantities take on the values

$$u_r^L = mn \frac{m + \alpha \sin 2\theta - \cos 2\theta}{\sqrt{1 + m^2 - 2m\cos 2\theta}}, \quad u_\theta^L = -n \frac{\alpha(1 - m\cos 2\theta) + m \sin 2\theta}{\sqrt{1 + m^2 - 2m\cos 2\theta}},$$

$$\nu^L = \frac{1 - \alpha^2}{1 + m^2 - 2m\cos 2\theta}, \quad P_{rr}^L = P_{\theta\theta}^L = 0, \quad P_{r\theta}^L = -\frac{2\alpha M}{1 + m^2 - 2m\cos 2\theta},$$

$$\Omega_{rr}^L = \Omega_{\theta\theta}^L = 0, \quad \Omega_{r\theta}^L = -\frac{m \sin 2\theta}{\sqrt{1 - \alpha^2(1 + m^2) - 2m(1 - \alpha^2)\cos 2\theta + m^2 \cos^2 2\theta}}.$$

The boundary displacements have the normal and tangential components $u_n = -u_r^L$ and $u_t = -u_\theta^L$. The magnitude of this displacement attains the following extreme values at the points of the ellipse that lie on the bisectrix of the coordinate angles:

$$\left|u^L\right|_{\max} = n(m + \alpha) \quad \text{for} \quad 2\theta = \frac{\pi}{2} \text{ and } \frac{5\pi}{2}, \qquad \left|u^L\right|_{\min} = n|m - \alpha| \quad \text{for} \quad 2\theta = \frac{3\pi}{2} \text{ and } \frac{7\pi}{2}.$$

The extreme values of the boundary relative density and stresses occur at the points on the symmetry axes of the ellipse and are equal to

$$\nu_{\max}^L = \frac{1 - \alpha^2}{(1 - m)^2}, \quad P_{r\theta\,\min}^L = -\frac{2\alpha M}{(1 - m)^2} \quad \text{for} \quad 2\theta = 0; 2\pi,$$

$$\nu_{\min}^L = \frac{1 - \alpha^2}{(1 + m)^2}, \quad P_{r\theta\,\max}^L = -\frac{2\alpha M}{(1 + m)^2} \quad \text{for} \quad 2\theta = \pi; 3\pi.$$

The averaged boundary rotations in the even and odd quarters of the ellipse have opposite directions and their extrema occur between the points on the symmetry axes:

$$\Omega_{r\theta\,\min}^L = -\frac{m}{\sqrt{1 - m^2 - \alpha^2}} \quad \text{for} \quad 2\theta = \arccos m, \ 2\pi + \arccos m,$$

$$\Omega_{r\theta\,\max}^L = \frac{m}{\sqrt{1 - m^2 - \alpha^2}} \quad \text{for} \quad 2\theta = 2\pi - \arccos m, \ 4\pi - \arccos m.$$

With allowance for relation (34) and the expression

$$a \cos^2 2\theta + 2b \cos 2\theta + c = (mr^2 \cos 2\theta + \alpha^2 r^4 - 1)^2 + \alpha^2 r^4 (1 - m^2 r^4 - \alpha^2 r^4) \quad (0 < r \leqslant 1),$$

the conditions of positive density and real rotation in the plate (33) constrain the parameter α and the boundary displacement (31):

$$\alpha^2 < 1, \qquad \alpha^2 < 1 - m^2.$$

Both inequalities are satisfied if the second inequality is satisfied. Thus, in comparison with the relative density, the averaged rotation imposes a stronger restriction on the boundary displacement.

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